An Overview of Existing Problems in Teaching the Science ''Fundamentals of Geometry''

Safarov Tulqin Nazarovich [1], and Abdullah Kurudirek [2]

[1] Department of Algebra and Geometry, Termiz State University, Termiz, Uzbekistan.
[2] Department of Mathematics Education, Tishk International University, Erbil, Iraq.
Correspondence: Abdullah Kurudirek, Department of Mathematics Education, Tishk International University, Erbil, Iraq.
Email: abdullah.kurudirek@tiu.edu.iq

DOI: 10.23918/ijsses.v11i2p91

Abstract: This paper goes into the challenges faced in the teaching of geometry, emphasizing its foundational principles. It investigates an alternate viewpoint by relating geometric principles to verses from the Holy Quran, implying that geometric conceptions have a spiritual and intellectual dimension. The historical contributions of Islam to geometric sciences are also examined, emphasizing the confluence between religion and geometry. The historical context of geometry in Islamic education is also explored, with an emphasis on the substantial contributions of some Muslim scholars to the topic between the 9th and 15th centuries. The literature overview presents much research on geometry education, including inquiry-based techniques, academic talent profiles, and the impact of various teaching methods on student achievement. Despite the variety of teaching methods, obstacles such as curriculum issues, teacher training, and student attitudes continue. In addressing the complexity of geometry teaching, the methodologies section highlights the significance of appropriate research design. The traditional teaching style and activity-based teaching/learning are addressed as two opposing methods. The latter is praised for its ability to foster innovative learning experiences. The results and discussion section critically assesses the "Foundations of Geometry" curriculum at top universities, identifying issues that need to be revised to line with contemporary expectations. The obstacles to teaching geometry are examined, including students' apathy and lack of prior knowledge, and solutions such as real-world examples, continual professional development, and activity-based teaching approaches are proposed. Finally, the article proposes a comprehensive reevaluation of geometry education that takes historical, religious, and current perspectives into account. It emphasizes the need for dynamic teaching methods, technology integration, and a revamped curriculum to make geometry more accessible and entertaining for students.

Keywords: Axiom, Affine, Curriculum, Distance, Euclidean Geometry, Measure, Non-Euclidean Geometry, Teaching.

1. Introduction

For decades, scholars have emphasized the importance of geometry instruction (Vigilante, 1967; Sinclair & Bruce, 2015; Franke & Reinhold, 2016). Engaging with geometric knowledge increases fundamental cognitive abilities, allowing for the development of specialized mathematical thinking strategies, and contributes significantly to comprehending our world (Bauersfeld, 1992).

Received: January 05, 2024
Accepted: April 29, 2024
Safarov, T. N., & Kuruderek, A. (2024). An Overview of Existing Problems in Teaching the Science "Fundamentals of Geometry". *International Journal of Social Sciences and Educational Studies, 11*(2), 91-106.

Furthermore, geometric ways of thinking pervade all mathematics since geometric thinking is important to this process whenever visual information is perceived, evaluated, and stored. Despite its acknowledged importance, geometry appears to have lost its place in school mathematics, being known as the "problem child" of mathematics education, as noted by (Backe-Neuwald, 2000) more than two decades ago.

Geometry, also known as the "science of space," plays a distinct and essential function in mathematics. Its principles and concepts are fundamental and essential to a wide range of real-world applications like architecture, engineering, design, and physics. Students must have a strong understanding of geometry to develop critical thinking, problem-solving skills, and spatial awareness. Nonetheless, despite its importance, teaching the principles of geometry has proven to be a complex and difficult endeavor.

This article presents a detailed summary of the current problems and challenges experienced in geometry teaching, particularly emphasizing the fundamental principles of this mathematical science. As educators and mathematicians work to improve their teaching techniques and approaches, it is vital to identify barriers to successful training. By throwing light on these concerns, we hope to foster a more nuanced understanding of the difficulties connected with teaching geometry, as well as a more informed discussion about potential solutions and advances in geometry education. Geometry is typically taught to children in their early years of school when they are exposed to fundamental shapes, angles, and measuring concepts. As students go through their schooling, they are presented with increasingly complex and sophisticated concepts, ranging from Euclidean geometry to more advanced, non-Euclidean models. This shift may be challenging, and many students struggle with the complexity of spatial thinking, frequently feeling disconnected from the subject's practical applications.

1.1 Another Perspective

Currently, we aim to delve into the incident from this particular perspective by opening another parenthesis. According to Muslim beliefs, the Holy Quran is the foundation of all sciences. In other words, enlightening avenues to those sciences can be discovered from this book. There are verses about the fundamentals of physics, astronomy, arithmetic, geometry, chemistry, biology, medicine, etc. For example, in the verse from "Surah Qamar" that says, "Certainly, we created everything (in an exact measure)," the word "measure" has two meanings: one carries the idea of "a specific limit" and the other conveys the meaning of "quantity", "measure." Both meanings are complementary. If we concentrate on the second sense, the concept of "measure" denotes the presence of a size or number for all things and occurrences relevant to all sciences.

The term "measure" in this sense indicates the existence of dimensions and quantities relevant to all disciplines of knowledge. Observing geometric dimensions is certainly straightforward here, but this verse, despite its basic lines, is miraculous in that it incorporates multiple meanings within itself. This verse is always being expanded with discoveries as science advances.

In other words, measurement is the act of comparing an object's qualities to those of measuring equipment (Van de Walle & Folk, 2005; Wijaya, 2009). Assume that length is a characteristic of an item that can be determined by measuring the distance between its ends. The Quran mentions a variety of measures, including distance, time, area, weight, and velocity. Additionally, the Quran guides operations requiring the use of specific units of measurement. The study of geometry concerning religion is known to have existed since ancient times, particularly through definitions given to geometric shapes.

For example, in geometry classes, a right-angled triangle was defined as "a geometric shape produced from Prophet Muhammad's (peace be upon him) prayer and the prostration of his servant," whereas in the 1865 book "Geometry," the first axiom was stated as "It is possible to draw only one straight line connecting two points." Because God created it that way," as stated in the content (Artikbayev & Xatamov, 2021). The utilization of mathematics in conjunction with the Quran regarding geometry and measurement materials has not been extensively applied (Huda, 2020).

The field of geometric sciences, also known as 'Ilm al-Handasa, flourished during the creative period spanning the 9th to 15th centuries. During this era, translated works from India and Greece were meticulously corrected and annotated, leading to significant contributions by Muslim scholars to the field of geometry. The noteworthy achievements of scholars such as Jawhari in the 9th century, Nayrizi in the 10th century, and Nasr ad-Din at-Tusi in the 13th century are just a fraction of the rich heritage of Islamic geometry (Shuriye & Daoud, 2011).

Geometry, more than any other branch of mathematics, has an impact on different sciences and technologies. The Holy Quran makes references to the concept of shapes, with a notable example being the Earth's shape. For centuries, many believed in a flat Earth, unaware of its spherical nature.

The father of algebra Al-Khawarizmi in the 9th century, a renowned mathematician, systematically applied algebra to geometry. Al Mahani, a contemporary of Al-Khawarizmi, proposed a method to transform geometric problems, like duplicating the cube, into algebraic ones. Another contemporary of his, the genius Thabit Ibn Qurra played a crucial role in laying the groundwork for significant mathematical discoveries, including analytic geometry and non-Euclidean geometry (Bakar, 2011).

As is obvious, we see the importance of recognizing the long-term contributions to the evolution of the science of geometry, which spans centuries, is an ongoing endeavor, and has emerged through numerous challenges. To maintain continuity with the past, we strive to provide a partial but necessary overview. We incorporate this viewpoint into our article is meant to broaden our understanding of the subject under consideration.

2. Literature Review

While conducting our research, we encountered numerous intriguing studies spanning various levels, some selection of which are outlined below.

Existing literature frequently suggests, sometimes with inadequate empirical support, that geometry is overlooked in the realm of mathematics education. This study specifically examines whether there have been any shifts in the neglect of geometry instruction over the past two decades, using two criteria adapted from (Ana Kuzle, 2022). The results support the hypothesis that, even after a paradigm change twenty years ago, geometry still gets less attention than other areas of mathematics. However, positive changes were observed when juxtaposed with the findings reported two decades earlier.

An inquiry-based approach to teaching geometry that begins with a puzzle-playing activity. This approach can help lower secondary school teachers gain a deep understanding of geometry in professional development courses. Another fascinating study that employed a qualitative research methodology revealed the geometry problem-solving profile of students using the field trip approach (Etika & Dewi, 2019). Researchers (Usiskin, 1982) looked into "Van Hiele's level and achievement in secondary school geometry." He developed a multiple-choice exam to determine the Van Hiele reasoning proficiency of a student. The goal was to determine whether these tests could in any way predict the geometry achievement of the students.

Genz (2006) investigated "Determining High School Geometry Students' Geometric Understanding Using van Hiele Levels." This research aimed to differentiate a curriculum that is nonstandards-based from that which is standards-based. Bhattarai (2005) conducted a research called "A Study on Problems Faced by Mathematics Students in Existing Curriculum." The study's goal was to identify the problems that pupils were experiencing with the present geometry curriculum.

Poudel's (2007) research "Problems Faced by Lower Secondary Mathematics Teachers in Teaching Geometry" sought to identify the challenges that lower secondary mathematics teachers encounter while teaching geometry. He found that teaching and learning geometry is ineffective because of the curriculum, textbooks, physical facilities, teaching-learning activities, resources, techniques, and student evaluation methods. He discovered that the bad attitude toward geometry is a psychological issue.

Chaudhary (2014) did a study titled "Difficulties faced by learning geometry at the lower secondary level." The purpose of this study was to discover the difficulties that pupils have when learning geometry at the lower secondary level. The study's findings point to a language discontinuity, a lack of adequate knowledge of geometry topics and figures, a lack of interpersonal interactions, no proper contact between teachers and students, poor class attendance, and a lack of language understanding in mathematics education. They used the experimental technique to conduct a study titled "Effectiveness of inductive method in teaching geometry at secondary level." The major purpose of this study was to compare student success in geometry taught using the inductive technique vs the deductive method.

No doubt that we are aware of several research demonstrating that geometry courses can be taught at various levels using various approaches and activities. Of course, more emphasis may be placed on teacher education so that all of this hard work is not in vain and is reflected in the field. Geometry centers can be established. (Mifetu, 2023) is an example of one of these studies. Teachers must create their tools to implement curriculum-based geometry teaching and learning. Tools should be carefully and methodically built such that teaching and learning are engaging, motivating, enjoyable, challenging, and productive.

The authors (Husnawati & Ikhsan, 2020) concluded that teaching and learning should also encourage students to actively participate and allow appropriate space for invention, creativity, and independence based on their abilities, interests, and physical development.

We also believe that several suggestions presented to solve the problems should be examined in the paper (Adolphus, 2011).

- Immediately arrange training sessions and seminars for mathematics teachers focusing on the effective teaching of geometry.
- Ensure the provision of essential infrastructure and facilities that can enhance the motivation for teaching and learning geometry.
- Integrate real-life situations into the curriculum to mitigate the abstract nature of the subject.
- Implement a reward system to motivate diligent teachers and students.

One of the current challenges we face involves issues associated with the utilization of technology (Serin, 2023). To effectively teach and learn geometry, students need to possess the ability to visualize, construct, and comprehend the formation of shapes in connection with relevant facts. Consequently, the integration of digital technologies becomes crucial in aiding students in visualization, observation, and understanding of these facts. G. Sketchpad, calculators, interactive whiteboards, and GeoGebra (Praveen & Kwan Eu, 2013) are among the digital tools available for geometry education. GeoGebra, in particular, stands out as dynamic geometric software that combines elements of statistics, calculus, algebra, geometry, arithmetic, and spreadsheets into a user-friendly package, facilitating mathematics learning and teaching at various levels (Abebayehu & Hsiu-Ling, 22021). Integrating technology into the teaching and learning process can help to improve geometry learning. In a technology-rich environment, students have the opportunity to explore, solve, and articulate geometric concepts in various ways. The incorporation of technology into the instruction of geometry contributes to a smoother learning experience (Sunzuma, 2023).

3. Research Methodology

3.1 Purpose and Approach

The major purpose of document analysis is to acquire a better knowledge of the challenges encountered when teaching geometry. We thoroughly study and assess important papers, such as textbooks, curricular guides, teaching materials, and research publications. Yıldırım (2010) defines qualitative research as "investigating and evaluating any event, phenomenon, or situation within its existing conditions." One of the qualitative research methods is document analysis, which is the examination and analysis of written materials about existing facts and events (Yıldırım & Şimşek, 2006). One of the qualitative research methods used in this study was the document analysis method. The data gathered through document review was analyzed using one of the data analysis methods, content analysis. The information gathered has been expressed in the article under various headings.

3.2 Document Selection

In our pursuit of enhancing geometry education, we diligently gather a variety of pertinent materials. Textbooks are great tools for understanding how geometric ideas are taught to students. In addition, we refer to Curriculum Guidelines, which specify the important subjects to be addressed in geometry lectures to provide thorough training. Looking further, research articles offer a scholarly viewpoint on effective teaching practices, allowing us to incorporate evidence-based methodologies into our educational approach. By focusing on papers from many sources and periods, we may broaden our understanding and change our techniques to match the changing environment of geometric education.

3.3 Document Appraisal

We thoroughly review all the documents we can access. First, we endeavor to investigate common themes, problems, and areas of concern in geometry education. Additionally, we take care to pay attention to terminology, instructional techniques, and recommended solutions.

Central Concepts in Teaching and Learning Geometry

For effective geometry instruction and to bring coherence to classroom activities, it is beneficial to consider and emphasize key ideas in geometry during your preparation and teaching. These encompass (Jones, 2002):

- Invariance: The study of qualities in a configuration that does not change during a series of transformations is how mathematician Felix Klein redefined geometry in 1872 (Kurudirek, 2023). Examples include propositions on invariance, such as Thales' theorem and theorems related to triangles. Identifying which properties are invariant can be challenging for students, and the utilization of dynamic geometry software can greatly assist in this aspect.
- Symmetry: Symmetry, a fundamental concept not only in geometry but across mathematics, is most immediately evident in geometry. In mathematics, symmetry is defined as a transformation of a mathematical object that retains certain properties. Symmetry is frequently employed to simplify and strengthen arguments.
- Transformation: Students can utilize transformation to build general notions of congruence and likeness and then apply them to different figures. Congruent figures, for example, are always connected by a reflection, rotation, slide, or glide reflection. Gaining a deeper comprehension of geometric connections requires a grasp of transformations.

3.4 Identifying Common Issues

During our rigorous analysis, we discovered numerous common issues impacting geometry instruction. One major difficulty is obsolete information in textbooks, where a lack of contemporary examples and modern applications of geometry might impede students' understanding. Furthermore, pedagogical gaps arise because curricular standards frequently omit essential areas or fail to prioritize the development of critical thinking abilities, possibly leaving students unprepared for real-world problem-solving. Furthermore, evaluation is a considerable problem, since research studies show challenges in effectively evaluating students' geometric knowledge. Furthermore, the essential component of student involvement must not be forgotten; we ask if students actively participate in geometry lessons and whether they believe the information is relevant to their lives since these aspects have a significant impact on the effectiveness of the learning experience.

In the following discussion, we will address two distinct approaches. The initial one serves merely informational purposes and is not advisable. The focal point, however, lies in the second method, which we aim to underscore and put into practice.

3.5 The Conventional Teaching Approach

The instructor is generally positioned as the dominant figure in the classroom, monitoring all activities and ensuring that all classroom knowledge flows through them using the deductive teaching technique. The conventional approach prioritizes content and keeps the instructor more active, subjective, and less emotionally engaged (Singh, 2004). It revolves around the rote memorization of factual information, often neglecting higher-order cognitive outcomes (Rao, 2001). This typical teaching style conflicts with the normal functioning of the human mind (Weber, 2006). Students are immersed in repetitive learning, and over extended periods of conventional teaching, it becomes challenging to sustain the interests and attention of learners (Cangelosi, 2006).

3.6 Activity-Based Teaching/Learning

The significance of learning through practical engagement is crucial for effective knowledge acquisition. Research indicates that the more senses are stimulated, the better an individual learns and the longer they retain information. Activities foster a sense of dynamism and intelligence among learners. Recognizing that education involves the comprehensive development of a child, it is imperative to incorporate various activities to enhance learners' personalities in diverse ways (Noreen & Rana, 2019).

The approach of activity-based instruction serves as a dynamic problem-solving method for learners. It enhances the creative aspect of learning, providing a tangible context for acquiring knowledge. Through diverse experiences, this method encourages the acquisition of information, skills, and values. It contributes to building students' self-confidence and understanding through hands-on experiences, fostering positive relationships and enthusiasm. When a child is allowed to explore independently in an optimal learning environment, the learning process becomes enjoyable and enduring.

Activity-based learning encourages learners to apply their innovative ideas, knowledge, and intellect to solve problems. This instructional approach places a central focus on the child, representing a child-centered methodology. It cultivates self-learning abilities among students, enabling them to learn at their own pace and according to their individual capabilities. As noted by (Johnson et al., 1998) the traditional model involved the teacher providing all resources to passive learners, whereas the innovative approach actively engages learners with resources and each other.

3.7 Evidence-Supporting Activity-Based Learning in Geometry Education

- Enhanced Conceptual Understanding: Chesnais's (2021) research revealed that students' conceptual understanding and problem-solving abilities are enhanced when they participate in practical exercises and use manipulatives during geometry instruction, as opposed to when they are taught using conventional methods.
- Increased Student Engagement: (JalanUdayana, 2017; Alrajeh & Shindel, 2020) found that activity-based geometry instruction increases student engagement, motivation, and interest in the topic, fostering a pleasant learning environment.
- Development of Spatial Reasoning Skills: Activity-based techniques, including geometric models, puzzles, and real-world applications, have been demonstrated to improve students' spatial reasoning abilities, which are crucial for geometric thinking and problem-solving.

- Promotion of Critical Thinking: As students are encouraged to investigate, evaluate, and apply geometric concepts in real-world situations, active learning practices in geometry education, according to (Bourn & Baxter, 2014) enhance higher-order thinking abilities.

4. Geometry in Higher Education

Methods and tools for teaching mathematics with other topics at higher education institutions have advanced in modern times. "Foundations of Geometry" is a subject taught in mathematics education departments of higher education institutions. This topic has evolved as the scientific foundation for non-Euclidean geometry. Furthermore, geometry is an ancient topic and is regarded as the first scientific subject taught to school children in its entirety. This is because, in geometry studies, pupils encounter scientific ideas such as "axioms," "theorems," and "proofs" for the first time. This geometry-specific condition is also pertinent to the subject of "Foundations of Geometry" in higher education institutions.

Understanding the subject of "Foundations of Geometry" is considered vital information for people who have earned skills in any branch of mathematics in the twenty-first century. Learning how Euclidean geometry was built in Euclid's book "Elements" was a huge feat for any aficionado, and the subject of "Foundations of Geometry" is currently regarded as a significant field for mathematicians. In the nineteenth and twentieth centuries, revolutionary changes occurred in every field. These transformations also caused changes in geometry. In addition to different non-Euclidean geometries, topology, theory of complexity, integral geometry, geometric calibration, and even discrete mathematics, often known as discrete geometry, have emerged in the modern era.

The projective metric geometry, defined by Keli and Kleyn in the projective metrical phases known as the "Erlangen program," is thought to be a scientific path influenced by V.I. Lobachevsky's idea (Кампо, 2014; Сосов, 2016). The number of $n-$dimensional projective metric geometries, according to the Keli-Kleyn theory, is 3^n (Розенфельд, 1969). So, in the plane, there are nine various geometries, one of which is Lobachevsky's geometry, which created alterations in the perspective of geometry and differs from Euclidean geometry, as well as seven different non-Euclidean geometries (Shuriye & Daoud, 2011). However, when the dimension of the phase space is greater than two, it is not difficult to imagine that the number of projective metrics in phase geometry becomes numerous.

The emergence of the concept of "manifolds" in the field, as well as the development of the topology section, which studies the characteristics of geometric shapes in continuous deformation, contributed to the creation of "manifolds" in various fields, not only in the field of mathematical sciences, but also in other specific sciences, and are widely used. The development of manifold theory and its application in numerous sectors has resulted in many results relating to this subject. However, according to V. Nesh's theory, any simple manifold of a large size can be created in the Euclidean phase. This is because constructing original manifolds is a scientific direction coming from the generalization of boundary theory in geometry. The theory of semi-Euclidean phases originated near the beginning of the twenty-first century, leading to the development of elliptic and hyperbolic phases and related geometries (Artikbayev & Safarov, 2023).

Recent developments in discrete mathematics, known as discrete geometry with graph theory, have given rise to a new path in geometry that is frequently used in current technology. These orientations, however, do not reflect the modern developments occurring in geometry as a whole, both in terms of the topic itself and its programs. Unfortunately, the subject "Fundamentals of Geometry," which was developed in the early twentieth century, has been retained without changing its essence up to the present day.

The subject of "Foundations of Geometry" has been taught based on the following topics:

- The "Elements" of Euclid and the Fifth Postulate. The appearance of non-Euclidean geometry. Axioms of Dependence and the Results They Produce.
- Sort Axioms. Results from the dependence and order axioms.
- Light Concept. Interpolation of points. Congruence axioms. Results derived from axiom groups 1-3.
- The Continuity Axiom. Measurement of segments and angles. The intersection of straight lines and circles. Axiom of parallelism.
- Triangle Equality. Axiom systems of Euclidean geometry in cartesian coordinate system.
- Consistency and completeness of Euclidean Geometry Axiom systems.
- Independence of Euclidean geometry axioms. Gilbert's axiomatization. Axiomatics in school textbooks.

The paper by J. Buda, titled "Integrating Non-Euclidean Geometry into High School" (Buda, 2017), explores the incorporation of non-Euclidean geometry into high school curricula. Furthermore, Judith N. Cederberg's publication, "A Course in Modern Geometries" (Cederberg, 2004), in conjunction with Francesco C., Boccaletti D., Roberto C's "The Mathematics of Minkowski Space-Time: With an Introduction to Commutative Hypercomplex Numbers" (Francesco et al., 2008), Wylie C. R's "Foundations of Geometry" (Wylie, 2009), Audun H's "Geometry: Our Cultural Heritage" (Holme, 2010), and Vincent & Athanase's (2019) "Eighteen Essays in Non-Euclidean Geometry", collectively discuss advancements in the domain of "Fundamentals of Geometry."

In the late nineteenth century, the investigation into the 'Fifth Axiom problem,' the 'Historical development of meta geometry,' 'Projective geometry,' 'Lobachevsky's geometric principles,' and 'Philosophical perspectives on meta geometry' in the writings of A. Bertrand and W. Russell formed part of the curriculum for the 'Fundamentals of Geometry' at Cambridge University." The research also looked into Kant's conception of 'metric geometries,' among other things (Bertrand, 2016).

Furthermore, while looking at the geometry courses given at the top 100 universities in developed countries, notably under the subject "Basics of Geometry," there is an emphasis on improving historical, philosophical, theological, and modern knowledge relating to geometry. Despite these attempts, it is worth noting that the existing curriculum in these schools does not thoroughly integrate modern geometric knowledge, showing a significant absence of its instruction in higher education.

5. Analysis and Discussion

When we examine the curriculum topics in "Foundations of Geometry" at some of the world's top universities, including the Massachusetts Institute of Technology, Stanford University, Cambridge University, the University of Oxford, Université de Paris, Sorbonne University, and the Humboldt University of Berlin. We discovered that new geometric concepts are being presented, such as:

- A brief history of meta geometry: a synthetic, metric, and projective study of the three phases of meta geometry's history.
- A collection of projective geometry axioms: generalization of projective geometry concerning Euclidean geometry, verification of the fifth postulate using diverse methods, and analysis of the social context of various times and nations.
- Using non-Euclidean geometry to describe the physical nature of modern scientists: projective and metric geometries, Kant's theory.
- Special geometry: the management of the properties of graphs created arbitrarily in one universal law, physical geometry, and Pangeometry.
- The fourth dimension in modern art and non-Euclidean geometry: 18 non-Euclidean geometry feasts.
- Minkowski phase and time mathematics: studying commutative hypercomplex numbers.
- Introduction to modern geometry: real-world dimensions, visual dimensions (mathematical concepts as a means of expressing geometric shapes), mathematical auxiliary dimensions, sociocultural dimensions (historical and developmental aims), and psychological geometry.

The curriculum and themes covered in the "Foundations of Geometry" course, in our opinion, need to be examined and altered to match contemporary demands and incorporate new geometric trends. As a result, it is critical to focus on the core structure of the subject during the revision process, allowing students to thoroughly understand the modern directions and their reciprocal relationships, as well as explain the general aspects and differences between them.

As a result, we have chosen the following topics to cover in the "Foundations of Geometry" course:

- The Fifth Postulate and Lobachevsky's Axiom: Historical difficulties concerning the fifth postulate, Lobachevsky's axiom, the axiomatic structure of the course, and the requirements placed on it, existing axiom systems.
- Linear and Affine Phases: Linear phase, quadratic form, linear transformations, affine phase, affine coordinate system, scalar multiplication.
- Modern Definition of Geometry: Concepts of distance and movement, Minkowskian uniqueness, Galilean uniqueness.
- Pseudo-Euclidean Phases: Three-dimensional affine phases, elliptic and hyperbolic geometries in three-dimensional affine phases, Keli-Kleyn and Poincaré interpretations of Lobachevsky's uniqueness.

Curricula should be formulated to guide educators. Furthermore, the implementation of curriculum practices is not always fully visible in the classroom setting, and the anticipated outcomes of change may not be realized immediately or may take longer to manifest than intended. In the present teaching landscape, it is imperative to tailor educational programs to individual characteristics to actively engage Generation Alpha and Generation Z in the learning process. The emerging generations Alpha and Z are believed to harbor distinct expectations from life compared to previous generations due to their upbringing in the technological era (Abylkassymova et al., 2022).

Learning Geometry involves exploring diverse representations, including virtual manipulatives, written mathematical formulas, and verbal explanations. These approaches aid students in constructing mathematical concepts and fostering critical thinking skills. However, students' lack of interest in the geometry component and their family backgrounds can impact their learning experiences in this subject. The outcomes of a teaching experiment reveal that student-centered learning approaches are more effective than traditional methods for teaching Geometry (Juman et al., 2022).

- A significant number of students exhibit disinterest in learning Geometry, often perceiving it as a challenging section within mathematics. The predominant use of traditional teaching methods by most instructors, coupled with insufficient time allocated to teaching geometry in the curriculum, results in inadequate practice and occasional neglect of the geometry section.
- Moreover, students often lack prior knowledge of Geometry due to insufficient practice with sample questions, and contemporary teaching tends to overlook the importance of emphasizing geometry.
- The method of teaching significantly influences students' Geometry learning outcomes. While many teachers adhere to traditional methods characterized by passive learning and teacher-centered approaches, activity-based teaching methods prove superior. The latter fosters active student participation, is primarily student-centered, and facilitates multi-way communication.

To address these challenges, it is suggested that teachers incorporate real-world examples when teaching Geometry concepts. Continuous professional development, such as visiting seminars to learn about new teaching strategies, including the use of computer applications for successful Geometry education, is encouraged. Introducing new Geometry topics by building on students' past knowledge through a variety of exercises is also encouraged. The teaching experiment demonstrates the effectiveness of activity-based teaching strategies for geometry training. Implementing activity-based teaching approaches not only helps students develop a good attitude toward geometry but also improves their capacity to solve everyday issues utilizing geometric knowledge. This technique not only increases students' interest in mathematics but also provides them with practical problem-solving abilities for everyday life.

6. Implications and Recommendations

After performing a comprehensive examination, we developed the following proposals to improve geometry instruction. First, we push for textbooks to be updated to reflect current research findings and real-world applications. This guarantees that students have access to the most relevant and current material in their subject of study. Second, we actively encourage professional development programs that prepare teachers for effective geometry education approaches. By providing educators with the necessary skills and methodologies, we can increase instruction quality and student understanding. Finally, we underline the need to implement student-centered techniques that promote active learning and problem-solving. Encouraging pupils to actively participate in their learning process leads to a stronger comprehension and recall of geometric ideas. Implementing these proposals will not only improve the quality of geometry education but will also better prepare students for their future academic and professional goals.

7. Conclusion

By covering these subjects in the "Foundations of Geometry" course, we intend to provide students with a solid grounding in foundational ideas pertinent to modern geometry. The curriculum's objectives are to familiarize students with the scientific writings and research done by mathematicians in the field of non-Euclidean geometry to solve real-world problems, foster problem-solving abilities, and instill a sense of mathematics's capacity to provide answers and be applied in real-world scenarios. To overcome the current issues in teaching "Foundations of Geometry," we propose the following recommendations:

- "Foundations of Geometry" can be included in the curriculum's mandatory core block of studies. This phase is critical since including this subject in the core block complies with the overall curriculum standards, ensuring that extra subjects enhance students' expertise in the field of bachelor's degree disciplines. This strategy assists in meeting students' aspirations for more knowledge in the context of the labor market, where standards for bachelor's degree programs and market demand for additional topics might alter rapidly.
- The subject's allotted hours can be expanded. Currently, the topic "Foundations of Geometry" is given more than a hundred hours on average, with half for lectures and half for individual study. However, given the requirement to cover crucial topics thoroughly, notably the fundamental notions of modern geometry, even this allocation may be insufficient. For example, introducing the concept that the number of projective metric geometries is 3^n when $n = 3$ indicates 27 different geometries necessitate more effort. Adjusting the allowed hours doubling will bring them in line with the targeted objectives. This change can be made over two semesters.
- Research, compare, and analyze the themes taught in the subject "Foundations of Geometry" at prestigious foreign universities. Furthermore, students should be given assignments for independent work on these issues that incorporate the best techniques and methodologies used internationally.
- Ensure that the inclusion of new subjects in the core block of subjects aligns with the overall curriculum requirements. It is critical to strike a balance between general education and specialized courses to equip students with a good foundation in both mathematics and natural sciences, as well as specialized disciplines.
- Develop the core interdisciplinary knowledge, practical skills, and talents required to deepen the subject's study and mastery. This includes improving logical thinking, reaching correct conclusions, increasing scientific literacy, phase thinking, and abstract reasoning. These qualities, which are developed from theoretical knowledge and experience, are required in all fields of human endeavor.

Studying geometry aids students in cultivating a range of skills, including visualization, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argumentation, and proof. Beyond mathematics, other fields like science, geography, art, design, and technology are also impacted by spatial reasoning. Engaging with tangible tools can also help with the development of fine motor skills.

Geometry is embedded in a culturally and historically meaningful setting, making it an excellent basis for mathematical inquiry. The topic reveals exciting, often surprising, or paradoxical insights that might motivate students to dive further into the "why" of mathematical ideas.

Educators may improve both students' learning experiences and general attitudes toward mathematics by teaching geometry in a way that piques their interest and stimulates investigation. Furthermore, geometry is an effective way for students to improve their visualization abilities, giving them a varied approach to handling mathematics and other issues that do not rely entirely on accurate diagrams or symbolic representation. In this approach, we help to shape individuals who will be important assets to society in the future.

Acknowledgment:

The authors are grateful to Dr. Ipek Saralar-Aras for her helpful discussion of the paper.

Conflict of interest:

There is no conflict of interest in this paper.

Author's Contribution:

We confirm that the manuscript has been read and approved by all named authors. We also confirm that each author has the same contribution to the paper.

References

Abebayehu, Y., & Hsiu-Ling, C. (2021). GeoGebra in mathematics education: a systematic review of journal articles published from 2010 to 2020. Interactive Learning Environments. https://doi.org/10.1080/10494820.2021.2016861

Abylkassymova, A., et al. (2022). Methods of teaching geometry in the framework of the updated curriculum in mainstream education. Cypriot Journal of Educational Science. 14(9): 3568-3577. https://doi.org/10.18844/cjes.v17i9.8135

Adolphus, T. (2011). Problems of Teaching and Learning of Geometry in Secondary Schools in Rivers State, Nigeria, Int. J. Emerg. Sci. 1(2), 143-152.

Alrajeh, T.S., & Shindel, B.W. (2020). Student Engagement and Math Teachers Support. Journal on mathematics education. 11(2): 167-180.

Ana Kuzle. (2022). The teaching of geometry in primary education: Is geometry still neglected in school mathematics? Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Bozen-Bolzano, Italy.

Artikbayev, A., & Safarov, T. (2023). “Geometriya asoslari” fanini o‘qitishga zamonaviy yondashish haqida. Fizika, matematika va informatika jurnal. №2.

Artikbayev, A., & Xatamov, I.(2021). Tekislikda to’qqiz geometriya, Toshkent; pp.154

Backe-Neuwald, D. (2000). Bedeutsame Geometrie in der Grundschule: Aus Sicht der Lehrerinnen und Lehrer, des Faches, des Bildungsauftrages und des Kindes [Significant geometry in primary school: From the viewpoint of the teachers, the subject, the educational mission, and the child] [Unpublished doctoral dissertation]. Universität Paderborn.

Bertrand, A., & W. Russell. (2016). Foundations of geometry. Cambridge at the University Press.

Bhattrai, T. (2005). A study of the problem faced by the secondary level mathematics students in existing curriculum. An unpublished master thesis, department of mathematics education, T.U.,Kirtipur.

Bakar, W.N.W. (2011). Mathematics in the Holy Quran. Journal of Academic Minds. Vol. 5, No. 1, 53-64.

Bauersfeld, H.(1992). Drei Gründe, geometrisches Denken in der Grundschule zu fördern [Three reasons to promote geometric thinking in elementary school]. In H. Schumann (Ed.), Beiträge zum Mathematikunterricht; (pp. 7–33). Franzbecker

Bourn, R., & Baxter, S.C. (2014). Creating Environments for Fostering Effective Critical Thinking in Mathematics Education (Math-EFFECTs). In 2014 ASEE Annual Conference & Exposition, pp. 24-332.

Buda, J. (2017). "Integrating Non-Euclidean Geometry into High School". Honors Thesis, pp-173.

Cangelosi, (2006). Quoted by National Science Foundation. Applications and Applied Mathematics (AAM): An International Journal, 1(1): pp. 62– 82.

Cederberg, J.N.(2004). A Course in Modern Geometries. Undergraduate Texts in Mathematics. Springer Science & Business Media.

Chaudhary, A. (2014). A study on causes of low achievement of dalit students. An unpublished Master's thesis. In Mathematics education. T.U. Kirtipur.

Chesnais, A. (2021). Enhancing classroom discourse about measure to foster a conceptual understanding of geometrical practices. ZDM–Mathematics Education. 53(2): 337-357.

Сосов, Е.Н. (2016). Геометрия Лобачевского и ее применение в специальной теории относительности: учеб.-метод. пособие / Е.Н. Сосов — Казань: Казан, ун-т, 84 с.

Etika, E.D. & Dewi, I.S. (2019). Profile of Students' Academic Skills on Geometry Problems Using Field Trip Method. The 2nd International Conference on Science, Mathematics, Environment, and Education, AIP Conf. Proc. https://doi.org/10.1063/1.5139754

Francesco C., Boccaletti, D., & Roberto, C. (2008).The Mathematics of Minkowski Space-Time: With an Introduction to Commutative Hypercomplex Numbers. Monograph. Frontiers in Mathematics.

Franke, M., & Reinhold, S. (2016). Didaktik der Geometrie in der Grundschule [Didactics of geometry in elementary school]. Springer Spektrum.

Genz, R. (2006). Determining high school geometry students' geometric understanding using van Hiele levels: Is there a difference between standards-based curriculum students and non-standards-based curriculum students? MA Brigham Young University, Provo.

Holme, A. (2010). Geometry: Our Cultural Heritage. Springer Berlin, Heidelberg, https://doi.org/10.1007/978-3-642-14441-7

Huda, N. (2020). Geometry and Measurement in Quran, Al-Jabar: Jurnal Pendidikan Matematika. Vol. 11, No. 2, Hal 307 – 316.

Husnawati, A., & Ikhsan, M. (2020). The efforts to develop the geometry teaching and learning tools based on acehnese culture through problem-based learning in junior high school students, IOP Conf.

Series: Journal of Physics: Conf. Series, 012056 IOP Publishing https://doi.org/10.1088/1742-6596/1470/1/012056

JalanUdayana, S. (2017). Creating Dynamic Learning Environment to Enhance Students' Engagement in Learning Geometry.

Johnson, D. W., Johnson R., & Smith K. (1998). Active Learning: Co-operation in the college classroom. Edina, MB:Interaction Book Co.

Jones, K. (2002). Issues in the Teaching and Learning of Geometry. In: Linda Haggarty (Ed), Aspects of Teaching Secondary Mathematics: perspectives on practice. London: Routledge Falmer, pp 121-139.

Juman, Z.A.M.S., et al. (2022). "Difficulties in Learning Geometry Component in Mathematics and Active-Based Learning Methods to Overcome the Difficulties." Shanlax International Journal of Education, 10(2): pp. 41–58. https://doi.org/10.34293/education.v10i2.4299

Кампо, Н. (2014). Пападопулос, А. "О так называемой неевклидовой геометрии Клейна" М.: ДМК Пресс; стр.91-136.

Kurudirek, A. (2023). Illuminating high school students with some interesting non-Euclidean geometries in the plane. International Journal of Statistics and Applied Mathematics, 8(4):63-68. https://doi.org/10.22271/maths.2023.v8.i4a.1100

Mifetu, R. K. (2023). Using activity method to address students' problem-solving difficulties in circle geometry. Contemporary Mathematics and Science Education. 4(1): ep23016. https://doi.org/10.30935/conmaths/13079

Noreen, R.& Rana, A.M.K. (2019). Activity-Based Teaching versus Traditional Method of Teaching in Mathematics at Elementary Level Bulletin of Education and Research, 41(2): pp. 145-159.

Praveen, S., & Kwan Eu, L. (2013). Effectiveness of Using GeoGebra on Students' Understanding in Learning Circles.Malaysian Online Journal of Educational Technology. 1(4): 1–11.

Poudel, D.P. (2007). A study on problem faced by lower secondary mathematics teachers in teaching geometry. An unpublished Master Thesis, Department of Mathematics Education, T.U. Kirtipur.

Розенфельд, Б.А. (1969). Неевклидовы пространства. М.: Наука.

Rao, D. (2001). Science education in developing countries. New Delhi; Discovery Publishing House, 124-126.

Serin, H. (2023). The Integration of Technological Devices in Mathematics Education: A Literature Review. International Journal of Social Sciences and Educational Studies. 10(3): 54-59. https://doi.org/10.23918/ijsses.v10i3p54

Singh, M. (2004). Modern teaching of Mathematics. New Delhi: Anmol Publications PVT. LTD.

Sinclair, N., & Bruce, C.D. (2015). New opportunities in geometry education at the primary level. ZDM Mathematics Education. 47: 319–329. https://doi.org/10.1007/s11858-015-0693-4

Shuriye, A.O., & Daoud, J.I. (2011). Islamic Mathematical Sciences. Australian Journal of Basic and Applied Sciences. 5(3): 51-59.

Sunzuma, G. (2023). Technology integration in geometry teaching and learning: A systematic review (2010–2022). LUMAT: International Journal on Math, Science and Technology Education. 11(3): 1–18. https://doi.org/10.31129/LUMAT.11.3.1938

Usiskin, Z. (1982). Van Hiele Level and Achievement in Secondary School Geometry. Final Report of the Cognitive Development and Achievement in Secondary School Geometry Project. Chicago: University of Chicago Press.

Van de Walle, J. & Folk, S. (2005). Elementary and Middle School Mathematics. Teaching Developmentally. Toronto: Pearson Education Canada Inc; 2005.

Vigilante, N. J.(1967). Geometry for primary children: Considerations. The Arithmetic Teacher. 14(6): 453–459.

Vincent, A., & Athanase, P.(2019). Eighteen Essays in Non-Euclidean Geometry. EMS IRMA Lectures in Math and Theoretical Physics.

Weber, E. (2006). Brain-based business. Retrieved from http://brainbasedbusiness.com.

Wijaya, A. (2009). Hypothetical Learning Trajectory dan Peningkatan Pemahaman Konsep Pengukuran Panjang. Prosiding Seminar Nasional Matematika dan Pendidikan Matematika Jurusan Pendidikan Matematika FMIPA UNY. hal 373-387

Wylie, C.R. (2009). Foundation of geometry. Textbook. Dover Publications.

Yıldırım, A., & Şimşek, H. (2006). Sosyal bilimlerde nitel araştırma yöntemleri (6. baskı). Ankara: Seçkin Yayıncılık.

Yıldırım, K. (2010). Nitel araştırmalarda niteliği artırma. İlköğretim Online. 9(1), 79-92.